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Tensor operators and Wigner-Eckart theorem for $\mathcal{U}_{q\to 0}(sl(2))$

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Abstract

Crystal tensor operators, which tranform under $\mathcal{U}_{q\to 0}(sl(2))$, in analogous way as the vectors of the crystal basis, are introduced. The Wigner-Eckart theorem for the crystal tensor is defined: the selection rules depend on the initial state and on the component of the tensor operator; the transition amplitudes to the states of the same final irreducible representation are all equal.

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Deformation of enveloping Lie algebra $\mathcal{U}_q(\mathcal{G})$ introduced by Drinfled-Jimbo [1], [2] is by now a subject of standard text book. For the arguments discussed in this paper see [3] where an accurate list of references can be found. In the limit $q \to 0$ it has been shown by M. Kashiwara [4] that $\mathcal{U}_q(\mathcal{G})$ admits a canonical peculiar basis, called *crystal basis*. Since that article crystal bases have been object of very intensive mathematical studies and have also been extended to the case of deformation of affine Kac-Moody algebras. However a point is still , to our knowledge, missing: it is possible to introduce the concept of q-tensor and q-Wigner-Eckart theorem in the limit $q \to 0$? Besides the mathematical interest, the question may be interesting in application in physics. It is clear that in this limit we are no more dealing with the deformation of an univeral enveloping Lie algebra, but it is interesting to study what are the relics of the symmetry structure described originally by the algebra $\mathcal{G}$ and then by the deformation of its enveloping algebra $\mathcal{U}_q(\mathcal{G})$. It is, indeed, well know that Wigner-Eckart theorem is one of the milestones in the application of algebraic methods in physics. Let us remark that one of the motivations to study the limit $q \to 0$ by Date, Jimbo and Miwa [5], which firstly discovered the peculiar behavior of n-dimensional $\mathcal{U}_{q\to 0}(gl(n,\mathcal{C}))$-modules, whose axiomatic settlement has been given in [4], was the study of solvable lattice models where the parameter q plays the role of the temperature. Moreover in [6] the quantum enveloping algebra $\mathcal{U}_q(sl(2) \oplus sl(2))$ in the limit $q \to 0$ has been proposed as symmetry algebra for the genetic code assigning the (4) nucleotides (elementary constituents of the genetic code) to the fundamental representation and the (64) codons (triplets of nucleotides) to the three-fold tensor product of the fundamental representation, using crystal basis.

In the following we will consider only the crystal basis for $\mathcal{U}_q(sl(2))$.

To set the notation, let us recall the definition of $\mathcal{U}_q(sl(2))$

$$[J_+,\, J_-] = [2J_3]_q \tag{1}$$

$$[J_3,\, J_\pm] = \pm\, J_\pm \tag{2}$$

where

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} \tag{3}$$

In the following we shall omit the lower label q

For later use let us remind that:

$$[n]_q! = [1]_q\, [2]_q \ldots [n]_q \tag{4}$$

The algebra $U_q(sl(2))$ is endowed with an Hopf structure. In particular we recall the coproduct is defined by

$$\begin{aligned} \Delta(J_3) &= J_3 \otimes \mathbf{1} + \mathbf{1} \otimes J_3 \\ \Delta(J_\pm) &= J_\pm \otimes q^{J_3} + q^{-J_3} \otimes J_\pm \end{aligned} \tag{5}$$

The Casimir operator can be written

$$C = J_+ J_- \, + \, [J_3][J_3 - 1] = J_- J_+ \, + \, [J_3][J_3 + 1] \tag{6}$$

For q generic, i.e. not a root of unity, the irreducible representations (IR) are lalelled by an integer or half-integer number j and the action of the generators on the vector basis $|jm>$, $(-j \leq m \leq j)$, of the IR is

$$J_3 \, |jm>= m \, |jm> \tag{7}$$

$$J_\pm \, |jm>= \sqrt{[j \mp m][j \pm m+1]} \, |j, m \pm 1>= F^\pm(j,m) \, |j, m \pm 1> \tag{8}$$

From eqs.(7)-(8) it follows

$$C \, |jm>= [j][j+1] \, |jm> \tag{9}$$

Let us study the behavior of a q-number $[x]$ for $q \to 0$. In the following the symbol $\sim$ in the equations has to be read equal in the limit $q \to 0$ modulo the addition of a function regular in $q = 0$. From the definition eq.(3) we have

$$[x]_{q \to 0} \, \sim \, q^{-x+1} \qquad\qquad x \neq 0 \tag{10}$$

So it follows that

$$F^\pm(j,m)_{q\to 0} \, \sim \, q^{-j+1/2} \tag{11}$$

$$[j][j+1]_{q \to 0} \, \sim \, q^{-2j+1} \tag{12}$$

$$[x]!_{q\to 0} \, \sim \, q^{-1/2 \, x\,(x-1)} \tag{13}$$

From eqs.(8) and (11) it follows that the action of the generator $J_\pm$ is not defined in the limit $q \to 0$. Let us define the element Γ_0 belonging to the center of $\mathcal{U}_q(sl(2))$

$$\Gamma_0 = C^{-1/2} \tag{14}$$

$$\Gamma_0 \, |jm>= ([j][j+1])^{-1/2} \, |jm>_{q\to 0} \sim \, q^{j-1/2} \, |jm> \tag{15}$$

Let us define

$$\widetilde{J}_\pm = \Gamma_0 \, J_\pm \tag{16}$$

These operators are well behaved for $q \to 0$. Their action in the limit $q \to 0$ will define the crystal basis:

$$\widetilde{J}_+ \, |jm> \;=\; |j, m+1> \quad \text{for } -j \leq m < j \tag{17}$$

$$\widetilde{J}_- \, |jm> \;=\; |j, m-1> \quad \text{for } -j < m \leq j \tag{18}$$

$$\widetilde{J}_+ \, |jj>= \widetilde{J}_- \, |j, -j>= 0 \tag{19}$$

The tensor product of two representations in the crystal basis is given by [4].

Theorem - If $\mathcal{B}_1$ and $\mathcal{B}_2$ are the crystal bases of the M_1 and M_2 $\mathcal{U}_{q\to 0}(sl(2))$-modules, for $u \in \mathcal{B}_1$ and $v \in \mathcal{B}_2$, we have:

$$\tilde{J}_-(u \otimes v) = \begin{cases} \tilde{J}_- u \otimes v & \exists\, n \geq 1 \text{ such that } \tilde{J}_-^n u \neq 0 \text{ and } \tilde{J}_+^n v = 0 \\ u \otimes \tilde{J}_- v & \text{otherwise} \end{cases} \tag{20}$$

$$\tilde{J}_+(u \otimes v) = \begin{cases} u \otimes \tilde{J}_+ v & \exists\, n \geq 1 \text{ such that } \tilde{J}_+^n v \neq 0 \text{ and } \tilde{J}_-^n u = 0 \\ \tilde{J}_+ u \otimes v & \text{otherwise} \end{cases} \tag{21}$$

So the tensor product of two crystal basis is a crystal basis and the states of the basis of the tensor space are pure states. In other words in the limit $q \to 0$ all the q-Clebsch-Gordan (q-CG) coefficients vanish except one which is equal to ± 1.. Let us recall the definiton of q-tensor for $U_q(sl(2))$ [7], [8], [9] and [3]. An irreducible q-tensor of rank j is a family of $2j+1$ operators T^j_m $(-j \leq m \leq j)$ which tranform under the action of the generators of $U_q(sl(2))$ as

$$q^{J_3}(T^j_m) \equiv q^{J_3}\, T^j_m\, q^{-J_3} = q^m\, T^j_m \tag{22}$$

or

$$[J_3,\, T^j_m] = m\, T^j_m \tag{23}$$

$$J_\pm(T^j_m) \equiv J_\pm\, T^j_m\, q^{J_3} \;-\; q^{-J_3 \pm 1}\, T^j_m\, J_\pm = F^\pm(j,m)\, T^j_{m\pm 1} \tag{24}$$

In deriving the above equations use has been made of the non trivial coproduct eq.(5). The q-Wigner-Eckart (q-WE) theorem now reads [10]

$$< JM|T^j_m|j_1 m_1 >= (-1)^{2j}\, \frac{< J||T^j||j_1 >}{\sqrt{[2J+1]}}\; < j_1 m_1 j m|JM > \tag{25}$$

where $< J||T^j||j_1 >$ is the reduced matrix element of the q-tensor T^j and $< j_1 m_1 j m|JM >$ is the q-CG coefficients. In the following we will use the explicit expression of the q-CG of [10]. It is useful rewrite the q-WE theorem eq.(25) in the following form

$$T^j_m\, |j_1 m_1 >= (-1)^{2j} \sum_{J=|j-j_1|}^{j+j_1} \frac{< J||T^j||j_1 >}{\sqrt{[2J+1]}}\; < j_1 m_1 j m|JM >\; |JM > \tag{26}$$

Our strategy to define the $(q \to 0)$-tensor and than the $(q \to 0)$-WE is the following:

1. Let us write eq.(24) in the form

$$J_\pm\, T^j_m\, q^{J_3} = q^{-J_3 \pm 1}\, T^j_m\, J_\pm \;+\; F^\pm(j,m)\, T^j_{m\pm 1} \tag{27}$$

   than we multiply both sides of eq.(27) from left and right by an element (not unique) Γ of the center of the algebra and define

$$\widehat{T}^j_m = \Gamma\, T^j_m\, \Gamma \tag{28}$$

Let us remark that $\widehat{T}^j_m$ is still a q-tensor operator of the same rank as T^j_m. Indeed it tranforms under the action of $J_{\pm,3}$ according to eqs.(22), (24) or (27) which have been derived by application of the coproduct eq.(5). We make the *conjecture* that an element Γ exists such that $\widehat{T}^j_m$ has a smooth and defined behaviour in the limit $q \to 0$. We will discuss below some explicit examples in which T^j_m is not defined in the limit $q \to 0$ and its reduced matrix element diverge, while on the contrary it is possible to define $\widehat{T}^j_m$ with a well defined limit.

2. We apply the $\widetilde{J}_\pm, J_3$ generators to eq.(26) written for $\widehat{T}^j_m$ for $j = 1/2$ and then we study the limit $q \to 0$, assuming that $< J||\widehat{T}^j||j_1 >$ has a well-defined behaviour in the limit.

3. From the study of 2) we deduce the action of the generators $\widetilde{J}_\pm, J_3$ in the limit $q \to 0$ on $\widehat{T}^{1/2}_m$

4. From the tensor product we can infer the action for the generic tensor..

To perform our second step we need to compute the $q \to 0$ limit of $< j_1 m_1 \frac{1}{2} m | JM >$. The result are reported in Table 1, where we have used the expressions of $< j_1 m_1 \frac{1}{2} m | JM >$ given in App.B of [10] and eq.(10).

J	m = 1/2	m = -1/2
$j_1 + 1/2$	$q^{j_1 - m_1}$	1
$j_1 - 1/2$	-1	$q^{j_1 - m_1}$ q

Table 1: Behaviour of the q-CG $< j_1 m_1 \frac{1}{2} m | JM >$ for $q \to 0$.

Using the results of Table 1, denoting by $\tau^{1/2}_m$ the q-tensor operator $\widehat{T}^{1/2}_m$ in the limit $q \to 0$ and

$$\left(\frac{< J||\widehat{T}^j||j_1 >}{\sqrt{[2J+1]}} \right)_{q \to 0} \sim < J||\tau^j||j_1 > \tag{29}$$

we get

$$\begin{aligned} \tau^{1/2}_{1/2} \, |j_1 m_1 > \; &= \; (-1) \, \delta_{j_1, m_1} \; < j_1 + 1/2 \, ||\tau^{1/2}||j_1 > \; |j_1 + 1/2, m_1 + 1/2 > \\ &+ \; < j_1 - 1/2 \, ||\tau^{1/2}||j_1 > \; |j_1 - 1/2, m_1 + 1/2 > \end{aligned} \tag{30}$$

$$\tau^{1/2}_{-1/2} \, |j_1 m_1 > = (-1) \; < j_1 + 1/2 \, ||\tau^{1/2}||j_1 > \; |j_1 + 1/2, m_1 - 1/2 > \tag{31}$$

Inspection of eq.(30) and eqÉ(31) shows that the r.h.s. of the equations has the structure of the tensor product of $\underline{\frac{1}{2}} \otimes \underline{j}$ in the crystal basis, see the above quoted Kashiwara's Theorem. Note that the **order of the tensor product** is important.

So we can write the action of the generators $\widetilde{J}_{\pm}, J_3$ on $\tau_m^{1/2}$, as:

$$J_3(\tau_m^{1/2}) \equiv m\,\tau_m^{1/2} \qquad \widetilde{J}_{\pm}\,(\tau_m^{1/2}) \equiv \tau_{m\pm 1}^{1/2} \tag{32}$$

Clearly, if $|m| > 1/2$ then $\tau_m^{1/2}$ has to be considered vanishing. Eq.(32) gives for the transformation of the tensor operator the same law as for the crystal basis. It has been proven by Rittenberg-Scheunert [9] that for quasitriangular Hopf algebra ($U_q(sl(2))$ is "almost" quasitriangular which does not affect the following considerations) the tensor product of tensor operators is a tensor operator. So by applying the Rittenberg-Scheunert's theorem and in the limit $q \to 0$ the Kashiwara's theorem we can extend eq.(32) to any value j. So we define *crystal tensor* of rank j a set of operator which transform under $\widetilde{J}_{\pm}, J_3$ according to eq.(32). As an explicit check and a further example, we compute the ($q \to 0$)-WE theorem for T^1. We need to compute the $q \to 0$ limit of $< j_1 m_1 1 m | JM >$. The result are reported in Table 2, where we have used the expressions of $< j_1 m_1 1 m | JM >$ given in App.B of [10] and eq.(10).

J	m = 1	m = 0	m = -1
j_1+1	$q^{2(j_1-m_1)}$	$q^{j_1-m_1}$	1
j_1	$-q^{-1}q^{j_1-m_1}$	$q^2 q^{2(j_1-m_1)} - 1 + \delta_{j_1 m_1}$	$q q^{j_1-m_1}$
j_1-1	1	$-q^{j_1-m_1}$	$q q^{2(j_1-m_1)}$

Table 2: Behaviour of the q-CG $< j_1 m_1 1 m | JM >$ for $q \to 0$.

Using the results of Table 2, we obtain in the limit $q \to 0$:

$$\begin{aligned} \tau_1^1\,|j_1 m_1> &= \; < J = j_1+1\,||\tau^1||j_1>\,|J, m_1+1> \qquad \text{if} \quad m_1 = j_1 \\ &= \; -< J = j_1\,||\tau^1||j_1>\,|J, m_1+1> \qquad \text{if} \quad m_1 = j_1 - 1 \\ &= \; < J = j_1-1\,||\tau^1||j_1>\,|J, m_1+1> \qquad \text{if} \quad m_1 < j_1 - 1 \end{aligned} \tag{33}$$

$$\begin{aligned} \tau_0^1\,|j_1 m_1> &= \; < J = j_1+1\,||\tau^1||j_1>\,|J, m_1> \qquad \text{if} \quad m_1 = j_1 \\ &= \; -< J = j_1\,||\tau^1||j_1>\,|J, m_1> \qquad \text{if} \quad m_1 < j_1 \end{aligned} \tag{34}$$

$$\tau_{-1}^1\,|j_1 m_1> = < j_1+1\,||\tau^1||j_1>\,|J, m_1-1> \tag{35}$$

Let us now proof the following statement:

Proposition 1 *If the q-tensors $\widehat{T}^{r_1}$ and $\widehat{T}^{r_2}$ have a well defined behaviour for $q \to 0$, i.e. the crystal tensors τ^{r_1} and τ^{r_2} are defined, than the q-tensors $\widehat{T}^R$, obtained by the tensor product of $\widehat{T}^{r_1}$ and $\widehat{T}^{r_2}$ has a well defined limit for $q \to 0$.*

Proof: Let us define

$$\widehat{T}_K^R = \sum_{k_1,k_2} < r_1 k_1 r_2 k_2 | RK > \widehat{T}_{k_1}^{r_1} \, \widehat{T}_{k_2}^{j_2} \tag{36}$$

Take the matrix element of the r.h.s. and l.h.s. of eq.(36) between the initial state $|j_1 m_1>$ and the final state $|JM>$. Insert the identity

$$\mathbf{1} = \sum_{j,m} |jm >< jm| \tag{37}$$

in the r.h.s. and apply the q-WE theorem eq.(25) for $\widehat{T}^{r_1}$ and $\widehat{T}^{r_2}$. We get

$$\begin{aligned} < JM| \, \widehat{T}_K^R \, |j_1 m_1 > = & \sum_{k_1,k_2,j,m} < r_1 k_1 r_2 k_2 | RK >< jm r_1 k_1 | JM >< j_1 m_1 r_2 k_2 | jm > \\ & \times \; \frac{< J \,||\, \widehat{T}^{r_1} \,||\, j >}{\sqrt{2J+1}} \; \frac{< j \,||\, \widehat{T}^{j_2} \,||\, j_1 >}{\sqrt{2j+1}} \end{aligned} \tag{38}$$

If we apply the q-WE to the l.h.s. of the above equation and make the limit $q \to 0$, as by assumption the r.h.s. of eq.(38) has a limit, it follows that

$$\left(\frac{< J||\widehat{T}^R||j_1 >}{\sqrt{[2J+1]}} \right)_{q \to 0} \sim < J||\tau^j||j_1 > \tag{39}$$

Use of eq.(37) requires at least a comment. The completeness of the basis $|jm>$ for $su(2)$ is a particular case of the completeness of the IRs of a compact group. For $q \neq 1$ we cannot appeal to this general property as we are no more dealing with a Lie group. However the completeness of the q-coherent states [11] for the q-bosons [12], [13] gives us an argument for the completeness of the states $|jm>$, as a realization of the deformed enveloping algebra $U_q(su(2))$, for q generic, and of its representations can be written in terms of q-bosons. See below for comments about the use of q-bosons in the $q \to 0$ limit. Let us remark that the knowledge of the elements Γ_{r_1} and Γ_{r2}, which allow to define respectively the crystal tensors τ^{r_1} and τ^{r_2} from q-tensors T^{r_1} and T^{r_2}, does not determine the element Γ_R, which allows to define the crystal tensor τ^R from the q-tensor T^R, obtained from the tensor product of T^{r_1} and T^{r_2}, as the elements of the center of the algebra do not commute with the generic q-tensors T^j, for $j \neq 0$.

Now let us discuss in some explict examples our *conjecture* that it is posssible to find an element Γ in the center of the algebra such that the operator $\Gamma \, T^j \, \Gamma$ is well defined in the limit $q \to 0$.

- Let us consider the vector operator constructed with the generators [10]

$$T^1_{\pm} = \pm \frac{1}{\sqrt{[2]}} \, q^{-J_3} \, J_{\pm} \tag{40}$$

$$
\begin{aligned}
T_0^1 &= \frac{1}{[2]}\,(q^{-1}[2J_3] \,+\, (q-q^{-1})\,J_+J_-) \\
&= \frac{1}{[2]}\,(q^{-1}[2J_3] \,+\, (q-q^{-1})\,(C-[J_3-1/2]^2)) \qquad (41)
\end{aligned}
$$

has no well defined meaning in the limit $q \to 0$. If we compute the reduced matrix element $< j_1||T^1||j_1 >$ (which is the only non vanishing) for the q-tensor eq.(41) from eq.(25) we get

$$
< j_1||T^1||j_1 >= \frac{\sqrt{[2j_1][2j_1+1][2j_1+2]}}{[2]} \qquad (42)
$$

and

$$
< j_1||T^1||j_1 >_{q\to 0}\sim\; q^{-3j_1+1} \qquad (43)
$$

Then, choosing $\Gamma = \sqrt{q^{1/2}\,\Gamma_0^3}$, we have from eqs.(14)-(12)

$$
< j_1||\widehat{T}^1||j_1 >_{q\to 0}\sim\; 1 \qquad (44)
$$

Let us remark that the moltiplication of Γ by a real number, the addition of any element of the center vanishing for $q \to 0$ as well as any functional construction of Γ_0 behaving in the limit $q \to 0$ as q^{-3j_1+1} does not modify our conclusion. Our choice is the *minimal* one.

- The $U_q(sl(2))$ can be realized in terms of q-bosons [12], [13] which have no well defined behaviour in the limit as it can be seen from the defining expression

$$
a_i a_j^+ - q^{\delta_{ij}}\,a_j^+ a_i = \delta_{ij} q^{-N_i} \qquad (45)
$$

$$
[N_i,\, a_j^+] = \delta_{ij} a_j^+ \qquad [N_i,\, a_j] = -\delta_{ij} a_j \qquad [N_i,\, N_j] = 0 \qquad (46)
$$

or from the relation between q-bosons and standard bosonic operators [14]. So we cannot extend the q-boson realization to the limit $q \to 0$. Using q-boson q-spinor operator have been constructed [7].

$$
T^{\frac{1}{2}}_{\frac{1}{2}} = a_1^+\,q^{\frac{N_2}{2}} \qquad T^{\frac{1}{2}}_{-\frac{1}{2}} = a_2^+\,q^{-\frac{N_1}{2}} \qquad (47)
$$

Howevere it is always possible to compute the q-spinor reduced matrix using eq.(47), even if in the limit $q \to 0$ the explicit realization of the tensor operator in terms of q-bosons is meaningless. Indeed in the case of the q-vector operator above defined, obviously we obtain the same result using the definition eq.(41) in terms of the abstract generators of $U_q(sl(2))$ or making use of the explicit realization of the algebra in terms of the q-bosons. From eq.(25) and the expression of q-CG we get

$$
< j_1 + 1/2\,||T^{\frac{1}{2}}||j_1 >= -\sqrt{[2j_1+1][2j_1+2]} \qquad (48)
$$

and

$$< j_1 + 1/2\,||T^{\frac{1}{2}}||j_1 >_{q\to 0}\sim\; q^{-2(j_1+1/4)} \tag{49}$$

Then, choosing $\Gamma = \sqrt{q}\,\Gamma_0$, we have from eqs.(14)-(12)

$$< j_1 + 1/2\,||\widehat{T}^{\frac{1}{2}}||j_1 >_{q\to 0}\sim\; -1 \tag{50}$$

If we use q-spinor operator hermitean conjugate to operator given in eq.(47), [3], which is

$$T^{\dagger,\frac{1}{2}}_{\frac{1}{2}} = -a_2\,q^{-\frac{(N_1+1)}{2}} \qquad T^{\dagger,\frac{1}{2}}_{-\frac{1}{2}} = a_1\,q^{\frac{(N_2+1)}{2}} \tag{51}$$

From eq.(25) and the expression of q-CG we get

$$< j_1 - 1/2\,||T^{\dagger,\frac{1}{2}}||j_1 >= -\sqrt{[2j_1][2j_1+1]} \tag{52}$$

and

$$< j_1 - 1/2\,||T^{\dagger,\frac{1}{2}}||j_1 >_{q\to 0}\sim\; q^{-2(j_1-1/4)} \tag{53}$$

Then, choosing $\Gamma = \sqrt{q}\,\Gamma_0$, i.e. the same value as for $T^{\frac{1}{2}}$ we have from eqs.(14)-(12)

$$< j_1 - 1/2\,||\widehat{T}^{\dagger,\frac{1}{2}}||j_1 >_{q\to 0}\sim\; -1 \tag{54}$$

In conclusion we have introduced ($q\to 0$)-tensor operators, which we call crystal tensor operators, making the conjecture that such operators can be obtained as the limit for $q\to 0$ of the q-tensor operator multiplied to the right and to the left by an element Γ of the center of the algebra. Let us emphasize that the choice of the element Γ is not unique. We have in some specific examples shown that our conjecture is realized and we have explicitly determined a (minimal up a factor) form of Γ. The transformation law for the generic crystal tensor operators is

$$J_3(\tau^j_m) \equiv m\,\tau^j_m \qquad \widetilde{J}_{\pm}\,(\tau^j_m) \equiv \tau^j_{m\pm 1} \tag{55}$$

Clearly, if $|m| > j$ then τ^j_m has to be considered vanishing. The ($q\to 0$)-Wigner-Eckart theorem can be written

$$\begin{aligned} \tau^j_m\,|j_1 m_1 > \;=\; & (-1)^{2j}\sum_{\alpha=0}^{2j} < j_1 + j + \alpha||\tau^j||j_1 > \;|j_1 + j + \alpha, m_1 + m > \\ & (\delta_{m_1,j_1-\alpha} + \delta_{-m,j-\alpha} - \delta_{m_1,j_1-\alpha}\,\delta_{m,j-\alpha}) \end{aligned} \tag{56}$$

Let us stress that while the q-WE theorem (for q generic) has the same form of the usual WE theorem, roughly speaking one has to replace the numerical expression by q-numerical expression, so its content (selection rules, relation between the transition amplitudes) is of the same form, the ($q\to 0$)-WE theorem has a completely different structure. The final IR depends not only from the rank of the tensor and initial IR, but in a crucial way from the initial state and from the component of the tensor in

m_1/m	1	0	-1
$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$
1	2	2	2
0	1	1	2
-1	0	1	2
$\frac{3}{2}$	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{5}{2}$
$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{2}$
$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$
$-\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$

Table 3: Selection rules for vector operator τ^1. In the entries m_1/m the value of final J in function of the components of τ^1, and of the initial state, for $\jmath_1 = \frac{1}{2}, 1, \frac{3}{2}$.

consideration. In Table 3 we report the selection rules for the case of a $(q \to 0)$-vector operator, for for $\jmath_1 = \frac{1}{2}, 1, \frac{3}{2}$.

In particular the highest weight state of the initial IR j_1 is always transformed under action of τ^j into a state of the final IR $J = j_1 + j$, while the lowest weight state is transformed into a state of any final IR (exactly one state if $j_1 \geq j$ with $J = j_1 + m$). Let us remark the peculiar feature that no vector crystal operator can be build up with the generators $\widetilde{J}_{\pm}, J_3$. Indeed such a vector crystal operator should connect any initial state to a state of the same IR and this is not the case as one can realize from the Table 3 or from the general form of the theorem eq.(56) The transitions between an initial state, belonging to IR j_1, and any final state, belonging to the IR J, are all equal.

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